Influence of the cosmological expansion on small systems

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Abstract. – The effect of the large-scale cosmological expansion on small systems is studied in the light of modern cosmological models of large-scale structure. We identify certain assumptions of earlier works which render them unrealistic regarding these cosmological models. The question is reanalyzed by dropping these assumptions to conclude that a given small system can experience either an expansion or a contraction of cosmological origin.

Since Hubble proposed his interpretation of the cosmological redshift as expansion of the Universe, there is an ongoing debate about what scales actually participate in this expansion, a problem which has been reconsidered by different authors since then (see Refs. [1–7], and further references therein). We will address the problem by relaxing some previous assumptions to get a model in better agreement with the current cosmological picture. We then find that the large-scale expansion does imply a (minute) time dependence of the volume of small systems, but also that this change can be, contrary to previous analysis, either an expansion or a contraction: the reason is that the origin of this effect is conceptually different from the origin of the effect studied in previous works.

Refs. [2–6] consider the simplified model of a small bound system embedded on a Robertson-Walker (RW) metric: we call this "the RW picture". By "small" is meant a size much shorter than the scales characteristic of the background RW metric, i.e., the horizon and the radius of curvature (e.g., it is customary to consider an atom, a planetary system or a galaxy). The effect of the background RW metric is then considered a small perturbation on the dynamical evolution of the small system, treated itself as a *test particle* independent of the RW background: Refs. [2,4] state that a small bound system suffers no expansion, because they simply consider the zeroth-order, unperturbed evolution; Refs. [3,5,6] compute the first correction, which yields an expansion of the system (albeit much slower than the cosmological expansion). However, the RW picture involves two important assumptions contrary to modern cosmological models: (i) via the Einstein equations, there must exist some sort of stress-energy source that can be considered uniformly distributed and Hubble-flowing even at the small scale of

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the bound system, (ii) the gravity pull exerted by this source must dominate over the self-gravity of the system, i.e., the effect of the system's stress-energy on the RW metric is a small perturbation. This latter assumption excludes the case in which the small system is bound by its own gravity, but it can be relaxed [3], as we shall describe below.

Let us examine if these assumptions are upheld by the prevailing cosmological models [8]. These models predict a "bottom-up" scenario of structure formation: due to the gravitational instability, the matter driving the cosmological expansion on the large scales ceases to Hubble-flow (it "decouples" from the expansion) and collapses, becoming inhomogeneously distributed below some scale R, which grows with time and today is certainly larger than the galactic scales. Thus the Hubble flow just plays the role of an initial condition which is gradually destroyed on ever larger scales by the gravitational instability. This result contradicts assumption (i). However, one cannot exclude the presence of a uniform and still Hubble-flowing background of relativistic particles (in Ref. [3] were mentioned photons, very light neutrinos and gravitational waves, but there can be others). Nevertheless, this background source, with density parameter $\Omega \ll 1$, does not fulfill the condition (ii): the source of self-gravity corresponds to a density parameter that ranges from $\Omega \sim 10^3 - 10^5$ for a galaxy to $\Omega \sim 10^{30}$ for a hydrogen atom.

This discussion clearly shows that the RW picture is only of academic interest and has little use for a realistic study of a possible small-scale expansion: the metric on small scales does not look at all like the large-scale RW metric, but rather corresponds to that of an inhomogeneous distribution of bound systems. In fact, the interest is shifting precisely towards an appreciation of this small-scale graininess: Whereas the old question was "how can there be constant-volume systems in an expanding Universe?", the new question is rather the reverse "how can there be a large-scale RW metric at all given the pervading small-scale graininess?". In other words, since the Einstein equations are nonlinear, it is not granted that they preserve their form upon coarse-graining and it can therefore be questioned whether the RW metric really represents the large-scale solution despite that the matter distribution looks homogeneous when smoothed on large scales [9]. In any event, we will not address this new question, and we confine ourselves to answer the old question with due consideration for the small-scale graininess.

In connection with these arguments, the results following from the RW picture can possibly be reinterpreted as an average: on the small scales, the cosmological expansion is only an average property of a grainy Universe and it could be perceived by the small system only by placing it at every location in the Universe. However, such interpretation is also unrealistic because the small system cannot be treated as a simple test particle, that can be anywhere: the small systems of interest belong always to high density collapsed regions (clusters, galaxies, gas clouds, ...). Thus, the averaging procedure should restrict the small system to being placed in these regions, which, according to the modern scenarios of structure formation, no longer expand. That is, the effect of the *local* environment on the small system will be unrelated to the universal expansion and the system will perceive this expansion, as Hubble himself did, only through its interaction with the receding distant matter, i.e., farther than the scale R mentioned above. Therefore, we consider the following simplified model, which, in view of the previous discussion, is more suitable than the RW picture: We begin with a set of pointlike masses (galaxies, say) homogeneously distributed and Hubble-flowing and make a spherical cavity of fixed radius R centered on one of these pointlike masses which is assumed to contain the bound system. Refs. [1, 2] consider this kind of model to conclude that the small system does not suffer any kind of expansion because the distribution of inhomogeneities outside the cavity has an effective spherical symmetry. However, *this* symmetry holds only on average, so that in principle there could be a nonvanishing effect on the small system due to the deviation from perfect isotropy. Our goal, therefore, is to study the influence of the (Hubble-flowing)

anisotropies on a small bound system and, in particular, whether they give rise to a change of its volume.

Let L denote the size of the small system and d_{H} the Hubble radius of the large-scale effective RW metric associated with the Hubble-flowing pointlike masses (galaxies). One then has the inequalities $L \ll R \ll d_{\mathrm{H}}$, representing that the system is small and that the Hubble flow is observable. The assumption of pointlike masses implicitly requires that their size is negligible compared to R. At this point, we introduce an important simplification: rather than carrying out a fully relativistic analysis, we will work in the Newtonian limit in the reference frame of the small bound system. This is justified by the above mentioned condition that d_{H} is much larger than the physically interesting length scales and by the assumptions that the involved velocities are non-relativistic and that there are no singularities in the metric. In fact, we are working within the same approximation customarily employed to study the formation of large-scale structure in cosmological models [8].

The fundamental quantity now is the gravitational potential $\phi(\boldsymbol{r}, t)$ created by the distant inhomogeneities. By virtue of the superposition principle of Newtonian gravity, we can add a uniform, Hubble-flowing matter distribution as another source of ϕ: this will allow us to consider also the RW picture, for comparison with previous works. The Newtonian equation of motion of the particles forming the small bound system is simply ($\boldsymbol{r}$ is the position of any one particle)

$$m \frac{\mathrm{d}^2 r_i}{\mathrm{d}t^2} = F_i(\boldsymbol{r}) - m \phi_{,i}(\boldsymbol{r}, t), \tag{1}$$

where F_i collects all the internal forces acting between particles within the system (including their mutual gravitational attraction), which keeps the system bound. The effect of the potential is considered as a perturbation over the evolution driven by the internal forces. Since the potential varies over length scales much larger than the size L of the system, one can Taylor-expand $\phi(\boldsymbol{r})$ around the coordinate origin $\boldsymbol{r} = \boldsymbol{0}$. The homogeneous gravitational acceleration term $\phi_{,i}(\boldsymbol{0})$ can be eliminated by taking the coordinate origin at the position of the free-falling center of mass of the system: then, the lowest order term in this expansion corresponds to the tidal stresses, $r_j \phi_{,ij}(\boldsymbol{0})$ (hereafter, we assume a summation over pairwise repeated indices) [10]. Furthermore, we can also assume that the characteristic time scales of the small system are much shorter than the cosmological scales of temporal variation of ϕ. This allows an adiabatic approximation, by which we can compute the response of the system as if the gravitational field were static and afterwards introduce its time dependence. Proceeding further requires a specification of the internal forces. We consider a many-particle system that can be modelled as a continuum and so Eq. (1) must be appropriately rewritten. This continuum model is an innovation over previous works [1–7]. While it somewhat simplifies the reasoning, it keeps the essential features, and the conclusions can be easily extrapolated to systems such as an atom or a planetary system.

We then consider the differential equation expressing equilibrium under the *small* deformations induced by the tidal forces [11],

$$\sigma_{ij,j}(\boldsymbol{r}) = \varrho_0(\boldsymbol{r}) r_j \phi_{,ij}(\boldsymbol{0}), \tag{2}$$

where $\sigma_{ij}(\boldsymbol{r})$ is the stress tensor field induced by the deformation and $\varrho_0(\boldsymbol{r})$ is the mass density field of the unperturbed body (density perturbations yield a term of second order in the perturbing tidal forces in Eq. (2)). Given that the deformation must be small, σ_{ij} will be a linear, local function of the displacement field $u_i(\boldsymbol{r})$, so that this equation reduces to an inhomogenous, linear equation for this field. A great simplification of this equation is achieved

by considering an elastic, isotropic solid, so that Hooke's law applies [11]:

$$\sigma_{ij} = \left(K - \frac{2}{3}\mu\right) u_{k,k}\delta_{ij} + \mu(u_{i,j} + u_{j,i}), \tag{3}$$

where K and μ are the elastic constants of the solid. Besides, we are only interested in the change of volume of the system due to the perturbing potential. Let V_0 denote the volume of the undeformed body, and V that after deformation. The volume change $\delta V \equiv V - V_0$ is the integral over the body of the field $u_{k,k}(\boldsymbol{r})$. This latter field can be shown to obey a Poisson Eq. by taking the i-derivative of Eq. (2) and using Eq. (3). This equation is easily solved with the natural boundary condition $u_k \equiv 0$ if $\phi \equiv 0$. One then finds $\delta V = C\mathsf{D}_{ij}\phi_{,ij}(\mathbf{0})$, with the proportionality factor $C = 1/(3K + 4\mu) > 0$ (basically the isothermal compressibility) and the second-rank tensor

$$\mathsf{D}_{ij} = -\frac{3}{4\pi}\int_{V_0} \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{r}' \,\frac{(\varrho_0(\boldsymbol{r})r_j)_{,i}}{|\boldsymbol{r} - \boldsymbol{r}'|} = -\frac{3}{4\pi}\int_{V_0} \mathrm{d}\boldsymbol{r}\,\varrho_0(\boldsymbol{r})r_j \int_{V_0} \mathrm{d}\boldsymbol{r}' \frac{\partial}{\partial r'_i}\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|}. \tag{4}$$

We further make the approximation of taking the body quasi-spherical (e.g., a star or a planet) to achieve mathematically simple expressions without sacrificing the physical content of the problem we are discussing. The $\boldsymbol{r}'$-integral is trivial for a spherical V_0, and one can show after some algebra that

$$\delta V = C\mathsf{I}_{ij}[\phi_{,ij}(\mathbf{0}) - \frac{1}{3}\phi_{,kk}(\mathbf{0})\delta_{ij}] - \frac{1}{6}C\mathsf{I}_{ii}\phi_{,jj}(\mathbf{0}), \tag{5}$$

where

$$\mathsf{I}_{ij} = \int_{V_0} \mathrm{d}\boldsymbol{r}\,(r^2\delta_{ij} - r_i r_j)\varrho_0(\boldsymbol{r}) \tag{6}$$

is the moment of inertia tensor of the undeformed body [12].

We have written the change of volume as the sum of the changes induced separately by the traceless tidal tensor and by its trace, respectively. The latter, because of the Poisson equation, is determined by the matter distribution at the position of the system (i.e., the smoothly distributed component in our case) and represents always a decrease of volume: the mass of the uniform component enclosed within the system causes an inward attraction which contracts the body with respect to its unperturbed state. The traceless tidal tensor, however, carries information about the distant matter distribution and does not produce a change of definite sign: it can represent either a contraction or an expansion; this will depend on the orientation of the body relative to the pointlike sources. Notice that this latter change does not arise either if the traceless tidal tensor vanishes (e.g., in the case of a perfectly isotropic matter distribution, as in the RW picture) or if the unperturbed body has *exact* spherical symmetry ($\mathsf{I}_{ij} \propto \delta_{ij}$).

The effect considered is static and no mention of an expanding Universe has been made so far. This enters through the time dependence of the tidal forces. The adiabaticity assumption implies that this temporal dependence is so slow that equilibrium, Eq. (2), holds at any instant. We can then just take expression (5) and include the time dependence of $\phi_{,ij}(\mathbf{0})$. A local expansion, if any, is measured by the time derivative of δV, thus comparing the volume of the body at successive instants, rather than with the volume of the unperturbed body. In the simple models we consider below, it is $\phi_{,ij}(\mathbf{0}, t) \propto a(t)^{-3}$, where $a(t)$ denotes the expansion factor, due to the dilution of the total (uniform + pointlike components) mass density by the Hubble flow. It then follows from Eq. (5) that $(\delta V)^{\cdot} = -3H\delta V$, where the dot denotes

temporal derivative and $H = \dot{a}/a$ is the Hubble function. Following [3], we can define a local, effective Hubble constant, $\tilde{H}$, as

$$\tilde{H} \equiv \frac{\dot{V}}{3V} = \frac{(\delta V)^{\cdot}}{3(V_0 + \delta V)} \approx -\frac{\delta V}{V_0} H. \tag{7}$$

This is the main result of the analysis: the effective Hubble constant is much smaller than the true Hubble constant in absolute value (because the deformation is small, $|\delta V| \ll V_0$), and it need not even be positive, indicating that the expanding Universe could give rise to a local contraction.

We now consider a couple of applications of the result (7). First, to compare with the RW picture, we disregard inhomogeneities altogether and take into account only the effect of a uniform component with mass density $\varrho_b(t) \propto a(t)^{-3}$. Then, one has $\phi(\boldsymbol{r}, t) = (2\pi/3)G\varrho_b(t)r^2$ and the tidal tensor $\phi_{,ij} = (4\pi G\varrho_b/3)\delta_{ij}$ is a completely local quantity. Hence [13]

$$\tilde{H}_{\mathrm{RW}} = \frac{2\pi C G \mathsf{I}_{ii} \varrho_b}{3V_0} H > 0. \tag{8}$$

Therefore, one has a local expansion, albeit much slower than the Hubble flow. For example, taking $\varrho_b \sim 10^{-26}\,\mathrm{kg\,m^{-3}}$ (corresponding to $\Omega \sim 1$) and values appropriate for the planet Earth ($M \sim 10^{24}\,\mathrm{kg}$, $R \sim 10^7\,\mathrm{m}$, and $C \sim 10^{-11}\,\mathrm{N^{-1}m^2}$ suitable for typical solids [14]), one finds $\tilde{H}_{\mathrm{RW}} \sim 10^{-30} H$. This conclusion agrees qualitatively with that obtained in Refs. [3,5,6] for a system consisting of two particles in a Keplerian orbit (the detailed numerical values depend on the particular system, i.e., on the typical value of the internal forces F_i in Eq. (1)). The result (8) could be reinterpreted considering ϱ_b to be the spatial average of the actual inhomogeneous density. But, as emphasized, the RW picture is inconsistent with present cosmological models. To address the effect of the cosmological expansion, one should rather use the more general result following from Eq. (5) and consider instead the effect of the traceless tidal tensor.

We study, therefore, the alternative model introduced before: there is no uniform component at all but only Hubble-flowing, pointlike masses beyond some *fixed* distance R from the small system. Thus, the density field observed by the small system can be written as $\varrho(\boldsymbol{r}, t) = \sum_\alpha M\delta[\boldsymbol{r} - a(t)\boldsymbol{r}^{(\alpha)}]$, with $|\boldsymbol{r}^{(\alpha)}| > R$. Here, M, $\boldsymbol{r}^{(\alpha)}$ refer to the mass and position at the present time of each pointlike source, respectively, while the scale factor $a(t)$ (with $a(\mathrm{now}) = 1$) accounts for the Hubble law. This density field then yields

$$\phi_{,ij}(\boldsymbol{0}, t) = \frac{G}{a(t)^3} \int_{x>R} \mathrm{d}\boldsymbol{x}\, \frac{\varrho(\boldsymbol{x}, t_{\mathrm{now}})}{|\boldsymbol{x}|^3} \left\{ \delta_{ij} - \frac{3x_i x_j}{|\boldsymbol{x}|^2} \right\}. \tag{9}$$

Since $\phi_{,ij}$ is now traceless, taking as coordinate axis the principal axis of the tensor I_{ij} and denoting by I_i the principal moments of inertia, Eqs. (5) and (7) yield

$$\tilde{H} = -\frac{C \sum_i I_i \phi_{,ii}(\boldsymbol{0})}{V_0} H. \tag{10}$$

This effective Hubble constant depends on the specific positions $\boldsymbol{r}^{(\alpha)}$ of the sources; an estimate of its typical value can be gained by averaging over different realizations of the distribution of sources *with the constraint that there is one at* $\boldsymbol{x} = \boldsymbol{0}$. Let $\langle \cdots \rangle_c$ denote this conditional average. We then have, assuming statistical homogeneity and isotropy, [8]

$$\begin{aligned} \langle \varrho(\boldsymbol{x}) \rangle_c &= \varrho_b [1 + \xi(|\boldsymbol{x}|)], \qquad (11)\\ \langle \varrho(\boldsymbol{x}) \varrho(\boldsymbol{y}) \rangle_c &= \varrho_b^2 [1 + \xi(|\boldsymbol{x}|) + \xi(|\boldsymbol{y}|) + \xi(|\boldsymbol{x} - \boldsymbol{y}|) + \zeta(|\boldsymbol{x}|, |\boldsymbol{y}|, |\boldsymbol{x} - \boldsymbol{y}|)], \end{aligned}$$

where we have identified the (unconditional) average with ϱ_b, the mean density driving the large-scale expansion, and ξ and ζ are, respectively, the two and three-point reduced correlation functions [15]. Expressions (9) and (10) now yield $\langle\tilde{H}\rangle_c = 0$. This result is not surprising: because of statistical isotropy in the distribution of distant matter, there is no effect at all on average, and so we recover the result in Refs. [1, 2]. However, for a *given* system at a *given* position, there will be a non-vanishing $\tilde{H}$, of which the order of magnitude can be estimated from its variance, $\langle\tilde{H}^2\rangle_c$. We define the ratio $\eta^2 = \langle\tilde{H}^2\rangle_c/(\tilde{H}_{\rm RW})^2$ as a measure of this effect relative to the prediction of the RW picture, Eq. (8). The exact expression for η follows from Eqs. (8-11):

$$\eta^2 = \frac{9}{(2\pi)^2}\int_{x,y>R} \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\, \frac{\xi(|\boldsymbol{x}-\boldsymbol{y}|)+\zeta(|\boldsymbol{x}|,|\boldsymbol{y}|,|\boldsymbol{x}-\boldsymbol{y}|)}{|\boldsymbol{x}|^5|\boldsymbol{y}|^5}(x_i^2\lambda_i)(y_j^2\lambda_j), \tag{12}$$

where $\lambda_i = (3I_i/\sum_k I_k) - 1$ is a measure of the departure of the unperturbed body's shape from a sphere. To estimate this ratio for a given body, we need a specific form of ξ and ζ. We can assume that galaxies are not-too-biased tracers of the total mass distribution and have approximately the same mass. The two-point galaxy correlation extracted from catalogs can be approximated by $\xi(r) \approx (r_0/r)^\gamma$, $\gamma \approx 1.7$ over a wide range of scales, from about 50 kpc up to 20 Mpc [8]. The scale r_0 physically represents the smoothing scale above which the smoothed density field of the pointlike sources appears homogeneous. In the standard cold dark matter (CDM) models, it is this large-scale homogeneous density which rules the cosmological expansion, and so one expects that the Hubble law holds also above this scale. And indeed, the standard conclusion from observations is that both r_0, R are of the order of the megaparsec [8]. Moreover, on large scales, $r > r_0 \approx R$, the correlations are small and asymptotically $1 \gg |\xi| \gg |\zeta|$, so it can be argued that the contribution of ζ to the integral (12) is at most of the same order as that of ξ. Therefore, under the assumption that the main contribution to the integral in Eq. (12) arises from the range of scales where ξ follows the power law, one can easily estimate the order of magnitude of η,

$$\eta^2 \sim \lambda^2 \left(\frac{r_0}{R}\right)^\gamma \int_{x,y>1} \frac{\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}}{|\boldsymbol{x}|^3|\boldsymbol{y}|^3|\boldsymbol{x}-\boldsymbol{y}|^\gamma}, \tag{13}$$

where $|\lambda|$ denotes the order of magnitude of λ_i, and the integral is convergent for $0 < \gamma < 3$ and of the order of unity. Consequently, if this result were extrapolated to a not-too-spherical body (with $|\lambda| \sim 1$, e.g., a rod or a disc), the local Hubble constant due to the distant sources would be of the same order as the prediction of the RW picture; the important difference is that $\tilde{H}$ is a fluctuating quantity that can take either sign. On the other side, for a (quasi-spherical) planet the local Hubble constant is considerably reduced. For example, regarding the Earth again, it is reduced by a factor $|\lambda| \sim 10^{-3}$ [12].

It has been recently claimed [16] that the power-law behavior holds over the whole range of scales probed by galaxy catalogs, and r_0 ($> 100\,\mathrm{h}^{-1}$Mpc) cannot be determined from the avalaible catalogs yet, so that we face what seems a fractal Universe. These authors, however, concede that the Hubble law already holds for R of the order of the megaparsec, so that $r_0 \gg R$, which constitutes what they call the *Hubble-de Vaucoleurs paradox.* If this interpretation proves right, then our previous reasoning would yield a value of η/λ much larger than unity (and dominated by the strong three-point correlation), so the local Hubble constant could therefore be much larger than the prediction according to the standard CDM models.

In conclusion, we have argued that the customary RW picture is not suitable to address the possible influence of the cosmological expansion on small systems. We have proposed a

conceptually new, improved picture which takes into account the observed small-scale inhomogeneity in the matter distribution and the fact that the *relevant* small systems cannot be considered simple test particles, since they are themselves part of the Universe and belong to collapsed structures. This has enabled us to analyze this effect within the framework of realistic cosmological models. The study has been carried out in the Newtonian limit of General Relativity, which already encompasses the correct physics of the problem and provides a clear and simple explanation of how the cosmological expansion may affect small systems. We deduce that no small-scale expansion occurs *on average* due to statistical isotropy of the matter distribution, but at a given position in space, tidal effects due to the distant, Hubble-flowing inhomogeneities may give rise either to an *expansion* or a *contraction* in time of the volume of a small system. This effect is, as expected, of a very small magnitude (e.g., for the Earth about 10^{33} times smaller than the cosmological expansion), so that, for all practical purposes, it can be safely ignored.

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